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On the Exactness of the Circular Complex

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INTRODUCTION

In this paper we are going to give an answer to the following problem:

Let A be any ring with identity and $M_n(A)$ the affine space of $n \times n$ matrices with entries in A. Consider the variety W of pairs (X, Y) of $n \times n$ matrices of $M_n(A)$ such that $XY = YX = 0_{n \times n}$.

Moreover, let

$$B = \frac{A[X_{ij}, Y_{ji}]}{I}, \qquad i, j = 1,..., n,$$

be the quotient of the polynomial ring $A[X_{ij}, Y_{ji}]$ and its ideal I generated by the elements

$$\sum_{k=1}^{n} X_{ik} Y_{kj}, \qquad i, j = 1,..., n,$$

$$\sum_{k=1}^{n} Y_{ik} X_{kj}, \qquad i, j = 1,..., n.$$

Finally consider the infinite complex of B-modules

$$\longrightarrow B^n \xrightarrow{\bar{X}} B^n \xrightarrow{\bar{Y}} B^n \xrightarrow{\bar{X}} B^n \xrightarrow{\bar{Y}} B^n \qquad (*)$$

where $\bar{X} = (\bar{X}_{ij})$, $\bar{Y} = (\bar{Y}_{ij})$ and $\bar{X}_{ij}$, $\bar{Y}_{ij}$ are the images in B via the quotient homomorphism of the elements X_{ij}, Y_{ij}.

We call (*) the "circular complex," and our claim is that it is exact.

Such result is achieved using one of the main statements in [1], which gives an explicit basis for $A[X_{ij}, Y_{ji}]/I$, i.e., the basis of the so-called "special standard double tableaux."

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In Section 1 we recall briefly how such basis of tableaux is obtained. In Section 2 we use the same basi to find an analogous one for the ring

$$C = \frac{A[X_{ij}, Y_{ji}, t]}{J},$$

where t is an indeterminate and J is the ideal generated by the polynomials

$$\sum_{k=1}^{n} X_{ik} Y_{ki} - t, \qquad \sum_{k=1}^{n} Y_{ik} X_{ki} - t, \qquad i = 1,..., n$$

$$\sum_{k=1}^{n} X_{ik} Y_{kj}, \qquad \sum_{k=1}^{n} Y_{ik} X_{kj}, \qquad i \neq j,\ i, j = 1,..., n.$$

As a matter of fact, such basis allows us to prove that t is not a zero divisor in C, which information gives a straightforward way of getting the requested exactness.

We point out that one cannot generalize such result to the case of the complex

$$\longrightarrow \tilde{B}^n \xrightarrow{\bar{X}^1} \tilde{B}^n \xrightarrow{\bar{X}^m} \tilde{B}^n \xrightarrow{\bar{X}^{m-1}} \tilde{B}^n \xrightarrow{\bar{X}^{m-2}} \tilde{B}^n \xrightarrow{\bar{X}^1} \tilde{B}^n \xrightarrow{\bar{X}^m} \tilde{B}^n \longrightarrow,$$

where $\tilde{B} = A[X_{ij}^{(s)}]/I$, $s = 1,..., m$; $i, j = 1,..., n$; $X^{(s)} = (X_{ij}^{(s)})$, and I is the ideal generated by the elements

$$\sum_{k=1}^{n} X_{ik}^{(i)} X_{kj}^{(i+1)}, \qquad \sum_{k=1}^{n} X_{ik}^{(m)} X_{kj}^{(1)}$$

and the $\bar{X}^{(s)}$ are the images via the quotient homomorphism of the $X^{(s)}$, as one can see even from the simplest example, i.e., the one given by the 1×1 matrices, $XY = YZ = ZX = 0$, since Z is clearly in the kernel of multiplication by Y but does not lie in the image of multiplication by X.

The author thanks David Eisenbud for suggesting this problem.

1. A Basis for $A[X_{ij}, Y_{ji}]/I$

We recall here briefly the results obtained in [1] which are going to be useful in the sequel.

Let A be any ring with identity and consider the variety W given in the Introduction. If V and U are free modules over A such that rank $V = n$, rank $U = n$, W represents in the affine space

$$\hat{A} = \mathrm{Hom}(U, V) \times \mathrm{Hom}(V, U)$$

the variety of pairs (φ_1, φ_2) of maps

$$\varphi_1 : V \to U, \qquad \varphi_2 : U \to V$$

such that

$$\varphi_1 \circ \varphi_2 = 0_U, \qquad \varphi_2 \circ \varphi_1 = 0_V.$$

If we choose bases $B(V) = \{a_1, \ldots, a_n\}$, $B(U) = \{b_1, \ldots, b_n\}$ for each module, we can identify the two varieties.

Now let A be the coordinate ring of the affine space A of dimension $2(n \times n)$, i.e., the polynomial ring $A[X_{ij}, Y_{ji}]$, $i, j = 1, \ldots, n$ and $B = A[W]$ the reduced coordinate ring of W and let I be the ideal in A generated by the elements

$$\sum_{k=1}^{n} X_{ik} Y_{kj}, \qquad \sum_{k=1}^{n} Y_{ik} X_{kj}, \qquad i, j = 1, \ldots, n.$$

In [1] it is shown that $\tilde{A}/I \cong B$ and that B has a basis formed by certain elements $T_{X,Y}$ which are called "special standard double tableaux" and which we are now going to describe.

Let the symbol

$$[i_1 \cdots i_s \mid \hat{j}_1 \cdots \hat{j}_{n-s}] \tag{**}$$

denote the determinant of the minor of the matrix $X = (X_{st})$ (resp. of the matrix $Y = (Y_{st})$), whose rows are those of indices $i_1, \ldots, i_s$ and whose columns are those whose set of indices is the complement $\{h_1 < \cdots < h_s\}$ taken in order, in $\{1, \ldots, n\}$ of the set of indices $\hat{J} = \{\hat{j}_1 \cdots \hat{j}_{n-s}\}$, times $(-1)^t$, t being the sign of the permutation $(\hat{j}_1 \cdots \hat{j}_{n-s} h_1 \cdots h_s)$.

In order to use the symbol (**) in an easy way, it is convenient to write it with the following compact expressions

$$[I \mid \hat{J}]_X, \qquad [I \mid \hat{J}]_Y,$$

where $I = (i_1 \cdots i_s)$ and $\hat{J} = (\hat{j}_1 \cdots \hat{j}_{n-s})$. Now let

$$T_X = (H_X \mid K_X) = (H \mid K)_X = \left(\begin{array}{l|l} i_{11} \cdots\cdots i_{1h_1} & j_{11} \cdots j_{1n-h_1} \\ i_{21} \cdots i_{2h_2} & j_{21} \cdots\cdots j_{2n-h_2} \\ \vdots & \vdots \\ i_{s1} \cdots i_{sh_s} & j_{s1} \cdots\cdots\cdots j_{sn-h_s} \end{array} \right),$$

$$T_Y = (H_Y \mid K_Y) = (H \mid K)_Y = \left(\begin{array}{l|l} i_{11} \cdots\cdots i_{1h_1} & j_{11} \cdots j_{1n-h_1} \\ i_{21} \cdots i_{2h_2} & j_{21} \cdots\cdots j_{2n-h_2} \\ \vdots & \vdots \\ i_{s1} \cdots i_{sh_s} & j_{s1} \cdots\cdots\cdots j_{sn-h_s} \end{array} \right),$$

where $1 \leqslant i_{rt} \leqslant n$ and $1 \leqslant j_{rt} \leqslant n$.

We can associate to $(H|K)_X$, $(H|K)_Y$ polynomials in $\tilde{A}$ which we shall write as

$$[i_{11}\cdots i_{1h_1}|j_{11}\cdots j_{1n-h_1}]_X\cdots[i_{s1}\cdots i_{sh_s}|j_{s1}\cdots j_{sn-h_s}]_X,$$

respectively,

$$[i_{11}\cdots i_{1h_1}|j_{11}\cdots j_{1n-h_1}]_Y\cdots[i_{s1}\cdots i_{sh_s}|j_{s1}\cdots j_{sn-h_s}]_Y.$$

The T_X (resp. T_Y) are called "standard" if both H_X and $\hat{K}_X$ (resp. H_Y and $\hat{K}_Y$), where

$$\hat{K}_X=\begin{pmatrix} j_{s1} & \cdots\cdots\cdots & j_{sn-h_s} \\ \vdots & & \\ j_{11}\cdots j_{1n-h_1} & & \end{pmatrix},\qquad \hat{K}_Y=\begin{pmatrix} j_{s1} & \cdots\cdots\cdots & j_{sn-h_s} \\ \vdots & & \\ j_{11}\cdots j_{1n-h_1} & & \end{pmatrix}$$

have rows that are strictly increasing sequences and columns that are non decreasing sequences.

We can now associate to T_X and T_Y the "special double tableau"

$$T_{X,Y}=\begin{pmatrix} \hat{K}' & \hat{K} \\ H & H' \end{pmatrix}_{\bar{s}},$$

where $\bar{s}=(s_1,s_2)$, $s_1=$ number of boxes in H, $s_2=$ number of boxes in K.

The $T_{X,Y}$ provide the above-mentioned basis. As a matter of fact in [1] it is shown that representation theory may be used to prove the linear independence of the $T_{X,Y}$ using the following facts:

(a) On the variety W we have a natural action of the group $G=Gl(n)\times Gl(n)$ defined as

$$\text{given } (X,Y)\in W,\qquad (g_0,g_1)\in G$$
$$(g_0,g_1)(X,Y)=(g_0Xg_1^{-1},g_1Yg_0^{-1}).$$

(b) It is a classical fact [2] that there is a 1–1 correspondence between the rational irreducible representations of $Gl(n)$ and Young diagrams s.t. the rows have length at most n; moreover, given a Young diagram of shape $\sigma=(\sigma_1\cdots\sigma_k)$, the dimension of the representation $\mathscr{L}(\sigma)$ corresponding to σ is given by the number of standard Young tableaux of shape σ filled with integers out of $1\cdots n$.

(c) Every irreducible representation for G is a tensor product of an irreducible representation for $Gl(V)$ times an irreducible representation for $Gl(U)$.

2. The Exactness of the Circular Complex

According to what was said in the Introduction, we are going to prove

Theorem 2.1. *The circular complex is exact.*

In order to achieve this goal, we are going to use

Lemma 2.2. *Let R be a ring, t an element in R which is not a zero divisor, X, Y $n \times n$ matrices with entries in R such that*

$$XY = YX = tI,$$

I being the $n \times n$ identity matrix.

Let $R' = R/\langle t \rangle$, and $\bar{X}, \bar{Y} \in M_n(R')$ be the matrices whose entries are the images via the quotient homomorphism of those in X, Y.

Clearly $\bar{X}\bar{Y} = \bar{Y}\bar{X} = 0$. Moreover the complex

$$R'^n \xrightarrow{\bar{X}} R'^n \xrightarrow{\bar{Y}} R'^n \xrightarrow{\bar{X}} R'^n \xrightarrow{\bar{Y}} R'^n$$

is exact.

Proof. Because of the symmetry between $\bar{X}$ and $\bar{Y}$, it is sufficient to prove that $\operatorname{Ker} \bar{Y} = \operatorname{Im} \bar{X}$.

We first notice that we have the canonical surjective homomorphism

$$\psi: R^n \to R'^n$$

such that the diagram

$$\begin{tikzcd} R^n \arrow[r, "X"] \arrow[d, "\psi"'] & R^n \arrow[r, "Y"] \arrow[d, "\psi"'] & R^n \arrow[d, "\psi"'] \\ R'^n \arrow[r, "\bar{X}"] & R'^n \arrow[r, "\bar{Y}"] & R'^n \end{tikzcd}$$

commutes. Now take $v \in \operatorname{Ker} \bar{Y}$. Let $w \in R^n$ be such that $\psi(w) = v$.

Clearly $Yw = tu$, $u \in R$. But then $tu = YXu$, and so $Y(Xu - w) = 0$. Because $XY = tI$ and as t by hypothesis is not a zero divisor, the multiplication by t is injective, therefore Y is injective too, so $w = Xu$. Reducing modulo $\langle t \rangle$ and writing $u' = \psi(u)$, we have that $v = Xu'$.

We now consider the ring C defined in the following way

$$C = A[X_{ij}, Y_{ji}, t]/J,$$

where J is the ideal generated by the polynomials

$$\sum_{k=1}^{n} X_{ik} Y_{ki} - t, \qquad \sum_{k=1}^{n} Y_{ik} X_{ki} - t, \qquad i = 1,..., n,$$

$$\sum_{k=1}^{n} X_{ik} Y_{kj}, \qquad \sum_{k=1}^{n} Y_{ik} X_{kj}, \qquad i \neq j, i, j = 1,..., n.$$

It is clear that if we consider the matrices $X, Y \in M_n(C)$ whose coefficients are the residue classes mod J of X_{ij}, Y_{ji}, we have

$$XY = YX = tI$$

and moreover $B = C/\langle t \rangle$ and $\bar{X}$, $\bar{Y}$ are obtained from X, Y reducing mod $\langle t \rangle$.

Therefore, because of Lemma 2.2, the proof of Theorem 2.1 is reduced to prove that t is not a zero divisor. This can be done using the standard basis recalled in Section 1.

To be more precise, we are going to prove

PROPOSITION 2.3. *If $T_{\bar{X},\bar{Y}}$ are the tableaux of the standard basis of B, then a basis for C is given by the elements of C $t^n T_{X,Y}$.*

Before proving Proposition 2.3, let us see how one gets immediately as a consequence

COROLLARY 2.4. *t is not a zero divisor in C.*

Proof. Let $c \in C$ be such that $tc = 0$, and let us write

$$c = \sum_{i=1}^{r} a_i t^{s_i} T^{(i)}_{X,Y}, \qquad a_i \in A;$$

then

$$0 = tc = \sum_{i=1}^{r} a_i t^{s_i + 1} T^{(i)}_{X,Y}.$$

As the elements

$$t^{s_i + 1} T^{(i)}_{X,Y}$$

are linearly independent, we have that $a_i = 0$ $\forall i$, therefore $c = 0$.

We now go back to the

Proof of Proposition 2.3. (a) Notice that if we give to t degree 2 and to X_{ij} and Y_{ji} degree 1, then the rings B and C are graded rings, because the ideal $\langle t \rangle$ is homogeneous and the quotient homomorphism is a homomorphism between graded rings.

Let c be an element in C. We want to show that such element c can be written as a linear combination of tableaux of type $t^n T_{X,Y}$. As such elements are homogeneous of degree $2^n + \deg T_{X,Y}$, we can clearly suppose that c is homogeneous.

We now use induction on the degree of c. If $\deg c = 1$, there is nothing to prove, because c is a linear combination of the X_{ij} and the Y_{ji}. So let us suppose that the statement is true for $n-1$. Let $\deg c = n$. But if this is the case, by Proposition 1.1 in [1], there exists a c_1 with $\deg c_1 = \deg c$, which is a linear combination of the elements $T_{X,Y}$ and is such that $c - c_1 \in \langle t \rangle$. But then $c - c_1 = tc_2$ and $\deg c_2 \nleq \deg c$. By induction

$$c_2 = \sum_{j=1}^{r} a_j t^{s_j} T^{(j)}_{X,Y}$$

so that

$$c = c_1 + \sum_{j=1}^{r} a_j t^{s_j+1} T^{(j)}_{X,Y}.$$

This gives our claim.

(b) We now prove linear independence of the elements $t^n T_{X,Y}$. First notice that the ideal J defines the variety $V \subset M_n(A) \times M_n(A) \times A$ consisting of triplets (M_1, M_2, z) with

$$M_1 M_2 = M_2 M_1 = zI;$$

I as before is the $n \times n$ identity matrix.

In order to prove linear independence, we can restrict ourselves to the case $A = \mathbb{Z}$. But such case is deduced from the case $A = \mathbb{Q}$, therefore we assume instead $A = \mathbb{Q}$.

On V there is a natural action of the group

$$G = Gl(n) \times Gl(n) \times Q^*$$

(Q^* being the multiplicative group of Q), defined in the following way:

$$\text{given } (g_1, g_2, \alpha) \in G \quad \text{and} \quad (M_1, M_2, z) \in V,$$

then

$$(g_1, g_2, \alpha)(M_1, M_2, z) = ((\alpha I)\, g_1 M_1\, g_2^{-1}, (\alpha I)\, g_2 M_2\, g_1^{-1}, \alpha^2 z).$$

But then by an argument analogous to the one given in [1, Proposition 1.2], we can, using the representation theory of the group $Gl(n) \times Gl(n) \times Q^*$, reduce ourselves to prove that the elements

$$t^n K_{X,Y} = t^n(K^{(1)} K^{(2)}),$$

where $k^{(1)}, K^{(2)}$ are the canonical tableaux, are non zero functions on V. As, if we compute $t^n K_{X,Y}$ in the point $[I, I, 1]$, I is the $n \times n$ identity matrix, we get the identity; our claim is proved.

Remarks. (1) The above argument clearly implies that the ideal J is radical and C is the reduced coordinate ring of V.

(2) Using the theory of Hodge algebras one can easily show using the results in [1] that V is a Cohen–Macaulay normal variety.

References

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2. H. Weyl, "The Classical Groups," Princeton Univ. Press, Princeton, N. J., 1946.